

Common Coupled Fixed Point Theorems for Maps in a T_0 -ultra-quasi-metric Space

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Abstract

In this article, we prove the existence of common fixed points for a pair of maps on a q -spherically complete T_0 -ultra-quasi-metric space. The present article is a generalization, in the asymmetric setting of the paper of Rao et al.[11]. The key point in the proof is the use of Zorn's lemma. We construct an appropriate chain and show that it has a maximal element, from which we extract the fixed point we are looking for. The choice of the sets, here open balls, is characteristic of this type of problems and the contraction condition are essential, specially when we are to establish the uniqueness of the fixed point.

Keywords-*q-spherical completeness; T_0 -ultra-quasi-metric space; common fixed point.*

1 INTRODUCTION

To prove results about fixed points or common fixed points for maps in T_0 -ultra-quasi-metric space satisfying some strict contractive conditions, supcontinuity of the maps is assumed as well as the space is required to be joincompact. It turns out that this is not the case for q -spherically complete T_0 -ultra-quasi-metric space. In this situation, supcontinuity of maps is not necessary to obtain fixed points. A fixed points theorem for maps in T_0 -ultra-quasi-metric space had been proved recently (compare [3]) and the result has been extended in [9] for multi-valued maps.

For recent results in the area of Asymmetric Topology, the reader is advised to consult [4, 5].

2 PRELIMINARIES

Definition 2.1 (Compare [2, page 2]) *Let X be a set and $d : X \times X \rightarrow [0, \infty)$ be a function mapping into the set $[0, \infty)$ of non-negative reals. Then d is an ultra-quasi-pseudometric on X if*

- (a) $d(x, x) = 0$ for all $x \in X$, and
- (b) $d(x, z) \leq \max\{d(x, y), d(y, z)\}$ whenever $x, y, z \in X$.

The conjugate d^{-1} of d where $d^{-1}(x, y) = d(y, x)$ whenever $x, y \in X$ is also an ultra-quasi-pseudometric on X .

If d also satisfies the following condition (known as the T_0 -condition):

(c) for any $x, y \in X$, $d(x, y) = 0 = d(y, x)$ implies that $x = y$, then d is called a T_0 -ultra-quasi-metric on X . Notice that $d^s = \sup\{d, d^{-1}\} = d \vee d^{-1}$ is an ultra metric on X whenever d is a T_0 -ultra-quasi-metric on X .

In the literature, T_0 -ultra-quasi-metric spaces are also known as non Archimedean T_0 -quasi-metric spaces. The set of open balls $\{ \{y \in X : d(x, y) < \epsilon\} : x \in X, \epsilon > 0 \}$ yields a base for the topology $\tau(d)$ induced by d on X .

Example 2.2 (Compare [13, Example 3]) Let $X = [0, \infty)$. Define for each $x, y \in X$, $n(x, y) = x$ if $x > y$, and $n(x, y) = 0$ if $x \leq y$. It is not difficult to check that (X, n) is a T_0 -ultra-quasi-metric space.

Notice also that for $x, y \in [0, \infty)$, we have $n^s(x, y) = \max\{x, y\}$ if $x \neq y$ and $n^s(x, y) = 0$ if $x = y$. The ultra metric n^s is complete on $[0, \infty)$ since n and n^{-1} are complete on $[0, \infty)$ (compare [2, Example 2]).

Definition 2.3 Let (X, d) and (Y, m) be two ultra-quasi-pseudometric spaces. We say that a map $f : (X, d) \rightarrow (Y, m)$ is supcontinuous if $f : (X, d^s) \rightarrow (Y, m^s)$ is continuous.

Let (X, d) be an ultra-quasi-pseudometric space. Let $x \in X$ and $r \in [0, \infty)$. By $C_d(x, r)$ we mean the closed ball

$$C_d(x, r) = \{y \in X : d(x, y) \leq r\}$$

of radius r centred at x .

Lemma 2.4 (Compare [2, Lemma 9]) If (X, d) is an ultra-quasi-pseudometric space and $x, y \in X$ and $r, s \in [0, \infty)$, then we have that

$$C_d(x, r) \cap C_{d^{-1}}(y, s) \neq \emptyset$$

if and only if

$$d(x, y) \leq \max\{r, s\}.$$

Definition 2.5 (Compare [2, Definition 2]) Let (X, d) be an ultra-quasi-pseudometric space. Let $(x_i)_{i \in I}$ be a family of points in X and let $(r_i)_{i \in I}$ and $(s_i)_{i \in I}$ be families of non-negative real numbers. We shall say that the family $(C_d(x_i, r_i), C_{d^{-1}}(x_i, s_i))_{i \in I}$ has the mixed binary intersection property provided that

$$d(x_i, x_j) \leq \max\{r_i, s_j\}$$

whenever $i, j \in I$.

Definition 2.6 We say that (X, d) is q -spherically complete provided that each family $(C_d(x_i, r_i), C_{d^{-1}}(x_i, s_i))_{i \in I}$ possessing the mixed binary intersection property also satisfies

$$\bigcap_{i \in I} (C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i)) \neq \emptyset.$$

For an example of a q -spherically complete ultra-quasi-metric space, the reader is advised to check [2, Example 2].

Proposition 2.7 (Compare [2, Proposition 2])

(a) Let (X, d) be an ultra-quasi-pseudometric space. Then (X, d) is q -spherically complete if and only if (X, d^{-1}) is q -spherically complete.

(b) Let (X, d) be a T_0 -ultra-quasi-metric space. If (X, d) is q -spherically complete, then (X, d^s) is spherically complete.

Definition 2.8 An ultra-quasi-pseudometric space (X, d) is called bicomplete provided that the ultra-pseudometric d^s on X is complete.

Proposition 2.9 (Compare [2, Proposition 3]) Each q -spherically complete T_0 -ultra-quasi-metric space (X, d) is bicomplete.

In [3], Agyingi proved the following:

Theorem 2.10 Let (X, d) be a q -spherically complete T_0 -ultra-quasi-metric space. If $T : X \rightarrow X$ is a mapping such that

$$d(Tx, Ty) < \max\{d(x, y), d(Tx, x), d(y, Ty)\},$$

for all $x, y \in X$, with $0 < \min\{d(x, y), d(y, x)\}$, then T has a unique fixed point. Now, we extend this theorem for a pair of maps of Jungck type.

3 Main Results

Definition 3.1 Let (X, d) be an ultra-quasi-pseudometric space, $f : X \rightarrow X$ and $T : X \rightarrow X$ be two self maps on X . We say that f and T coincidentally commute at $z \in X$ if $Tfz = fTz$.

Theorem 3.2 Let (X, d) be a q -spherically complete T_0 -ultra-quasi-metric space. If f and T are self map on X satisfying

$$T(X) \subseteq f(X), \tag{1}$$

$$d(Tx, Ty) < \max\{d(fx, fy), d(Tx, fx), d(fy, Ty)\}, \tag{2}$$

for all $x, y \in X$, with $0 < \min\{d(x, y), d(y, x)\}$, then there exists $z \in X$ such that $fz = Tz$.

Further, if f and T coincidentally commute at z , then z is the unique fixed point of f and T .

Proof.

Let $a \in X$. Let us denote by

$$C_d^a = C_d(fa, \alpha_a) \text{ and } C_{d^{-1}}^a = C_{d^{-1}}(fa, \alpha_a),$$

with $\alpha_a := d^s(fa, Ta) = \max\{d(Ta, fa), d(fa, Ta)\}$ and set

$$C^a = C_d^a \cap C_{d^{-1}}^a.$$

Let $\mathcal{A} := \{C^a : a \in X\}$. Define the relation \preceq on \mathcal{A} by

$$C^a \preceq C^b \text{ if and only if } C^b \subseteq C^a.$$

Then (\mathcal{A}, \preceq) is a partially ordered set. Since the verification of this fact is trivial, we leave it to the interested reader.

Let \mathcal{A}_1 be a nonempty chain in \mathcal{A} . Then by q -spherical completeness of (X, d) , we have that

$$\bigcap_{C^a \in \mathcal{A}_1} C^a = C \neq \emptyset.$$

Clearly, $C \cap f(X) \neq \emptyset$. It follows from the nature of the balls considered, the centers are of the form $fa, a \in X$. Moreover, we consider a totally ordered family of such balls. In fact,

$$\bigcap_{C^a \in \mathcal{A}_1} C^a = C^y$$

for some $y \in X$. For more clarity in this part, we refer the reader to [11].

Let $fb \in C$ and $C^a \in \mathcal{A}_1$. Then we have

$$d(fa, fb) \leq \alpha_a \text{ and } d(fb, fa) \leq \alpha_a.$$

Let now $x \in C^b$. Then

$$d(fb, x) \leq \alpha_b \text{ and } d(x, fb) \leq \alpha_b.$$

$$\begin{aligned} d(fb, x) &\leq \max\{d(Tb, fb), d(fb, Tb)\} \\ &\leq \max\{d(Tb, Ta), d(Ta, fa), d(fa, fb), d(fb, fa), d(fa, Ta), d(Ta, Tb)\} \\ &\leq \max\{d(Tb, Ta), d(Ta, Tb), \alpha_a\} \\ &\leq \max\{d(Tb, fb), d(fb, fa), d(fa, Ta), d(Ta, fa), d(fa, fb), d(fb, Tb)\} \\ &\leq \alpha_a \end{aligned}$$

From the above inequality, we have now that

$$\begin{aligned} d(fa, x) &\leq \max\{d(fa, fb), d(fb, x)\} \\ &= \alpha_a \end{aligned}$$

which means that $x \in C_d(fa, \alpha_a)$. We have thus shown that

$$C_d(fb, \alpha_b) \subseteq C_d(fa, \alpha_a). \quad (3)$$

By a similar computation, one can show that

$$C_{d-1}(fb, \alpha_b) \subseteq C_{d-1}(fa, \alpha_a). \quad (4)$$

By Equations (3) and (4), we have that for all $C^a \in \mathcal{A}_1$, $C^b \subseteq C^a$, which means that $C^a \preceq C^b$ for all $C^a \in \mathcal{A}_1$. Thus C^b is an upper bound in \mathcal{A} for the chain \mathcal{A}_1 . By Zorn's lemma we conclude that \mathcal{A} has a maximal element, say, C^u , $u \in X$. We claim that $Tu = fu$.

Suppose on the contrary that $fu \neq Tu$. Since $Tu \in T(X) \subseteq f(X)$, there exists $w \in X$ such that $Tu = fw$. From (2), we have

$$\begin{aligned} d(fw, Tw) &= d(Tu, Tw) \\ &< \max\{d(fu, fw), d(Tu, fu), d(fw, Tw)\} \\ &< \max\{d(fu, fw), d(fw, fu), d(fw, Tw)\} \\ &= \max\{d(fu, fw), d(fw, fu)\}. \end{aligned}$$

and

$$\begin{aligned}
d(Tw, fw) &= d(Tw, Tu) \\
&< \max\{d(fw, fu), d(fu, Tu), d(Tw, fw)\} \\
&< \max\{d(fw, fu), d(fu, fw), d(fw, Tw)\} \\
&= \max\{d(fu, fw), d(fw, fu)\}.
\end{aligned}$$

Thus $fu \notin C^w$ and hence $C^u \subsetneq C^w$. It is a contradiction to the maximality of C^u . We therefore conclude that $Tu = fu$.

Moreover, if f and T coincidentally commute at u , then $f^2u = f(fu) = f(Tu) = T(fu) = T^2u$.

Suppose that $fu \neq u$. From condition (2), we have that

$$\begin{aligned}
d(Tfu, Tu) &< \max\{d(f^2u, fu), d(Tfu, f^2u), d(fu, Tu)\} \\
&= d(f^2u, fu) = d(Tfu, Tu),
\end{aligned}$$

since $d(Tfu, f^2u) = d(fu, Tu) = 0$.

Similarly we can prove that $d(Tu, Tfu) < d(Tu, Tfu)$. The above inequalities gives a contradiction and so we must have that $u = fu = Tu$.

Let us now prove uniqueness.

Suppose that there is another common fixed point, i.e., there exists $z \in X$ such that $f(z) = z = Tz$ and $z \neq u$. We shall examine two cases.

Case 1: Suppose $d(z, u) > 0$. Then we have that

$$d(z, u) = d(Tz, Tu) < \max\{d(fz, fu), d(Tz, fz), d(fu, Tu)\} = d(z, u),$$

which is a contradiction.

Case 2: Suppose now that $d(u, z) > 0$. Then we get

$$d(u, z) = d(Tu, Tz) < \max\{d(fu, fz), d(u, z), d(fz, Tz)\} = d(u, z),$$

which is a contradiction.

Thus we must have that $z = u$.

□

Corollary 3.3 *Theorem 3.2 holds if inequality (2) is replaced by*

$$d(Tx, Ty) < \max\{d(Tx, Ty), d(Tx, fx), d(fy, Ty), d(fx, Ty), d(Tx, fy)\}, \quad (5)$$

for all $x, y \in X$, with $0 < \min\{d(x, y), d(y, x)\}$.

Proof.

Since $d(fx, Ty) \leq \max\{d(fx, fy), d(fy, Ty)\}$ and $d(Tx, fy) \leq \max\{d(Tx, fx), d(fx, fy)\}$, hence inequality (5) implies inequality (2).

□

Corollary 3.4 *Taking $f = I$ (the identity map), we obtain Theorem 1 of [3].*

4 MULTI-VALUED MAPS

Now we generalize Theorem 3.2 when T is a multi-valued map.

Let $\mathcal{P}_0(X) := 2^X \setminus \emptyset$ where 2^X denotes the power set of X . For $x \in X$ and $A, B \in \mathcal{P}_0(X)$, we set:

$$d(x, A) = \inf\{d(x, a), a \in A\} \text{ and } d(A, x) = \inf\{d(a, x), a \in A\},$$

and define the Hausdorff ultra-quasi-pseudometric H by

$$H(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b) \right\}.$$

Then H is an extended ultra-quasi-pseudometric on $\mathcal{P}_0(X)$. Moreover, we know from [?] that the restriction of H to $S_{cl} = \{A \subseteq X : A = cl_{\tau(d)} A \cap cl_{\tau(d^{-1})} A\}$ is an extended T_0 -ultra-quasi-metric.

Definition 4.1 For a non Archimedean T_0 -quasi-metric space (X, d) , we denote by 2_j^X the space of all nonempty joincompact subsets in X with the Hausdorff ultra-quasi-metric H .

We have the following result due to Aggyingi [3].

Theorem 4.2 (compare [3]) Let (X, d) be a q -spherically complete T_0 -ultra-quasi-metric space. If $T : X \rightarrow 2_j^X$ is a mapping such that

$$H(Tx, Ty) < \max\{d(x, y), d(Tx, x), d(y, Ty)\} \text{ for all } x, y \in X, x \neq y,$$

then T has a unique fixed point, i.e there exists $x \in X$ such that $x \in Tx$.

Definition 4.3 Let (X, d) be an ultra-quasi-pseudometric space, $f : X \rightarrow X$ and $T : X \rightarrow 2_j^X$. f and T are said to be coincidentally commuting at $z \in X$ if $fz \in Tz$ implies that $fTz \subseteq Tfz$.

Theorem 4.4 (compare [11, Theorem 9]) Let (X, d) be a q -spherically complete T_0 -ultra-quasi-metric space. Let $f : X \rightarrow X$ and $T : X \rightarrow 2_j^X$ be two maps satisfying

$$Tx \subseteq f(X) \quad \forall x \in X, \quad (6)$$

$$H(Tx, Ty) < \max\{d(fx, fy), d(Tx, fx), d(fy, Ty)\}, \quad (7)$$

for all $x, y \in X$, with $0 < \min\{d(x, y), d(y, x)\}$. Then there exists $z \in X$ such that $fz \in Tz$.

Further, assume that

$$d^s(fu, fx) \leq \min\{H(Tu, Tfy), H(Tfy, Tu)\} \quad \forall x, y, u \in X \text{ with } fx \in Ty, \quad (8)$$

and

$$f \text{ and } T \text{ are coincidentally commuting at } z. \quad (9)$$

Then fz is the unique fixed point of f and T .

Proof.

Let $a \in X$ and denote by

$$C_a^d = C_d(fa, \beta_a) \text{ and } ; C_a^{d^{-1}} = C_{d^{-1}}(fa, d(fa, Ta))$$

the closed balls with centers at $fa \in X$ and radius

$$\beta_a := \max\{d(Ta, fa), d(fa, Ta)\}$$

with

$$d(Ta, fa) = \inf\{d(z, fa) : z \in Ta\}$$

and

$$d(fa, Ta) = \inf\{d(fa, z) : z \in Ta\}.$$

Put

$$C_a = C_a^d \cap C_a^{d^{-1}}.$$

Let \mathcal{A} be the collection of all such closed balls C_a such that a runs over X . Define \preceq on \mathcal{A} by

$$C_a \preceq C_b \text{ if and only if } C_b \subseteq C_a.$$

Then (\mathcal{A}, \preceq) is a partially ordered set. We leave the verification of this fact to the interested reader.

Let \mathcal{A}_1 be a nonempty chain in \mathcal{A} . Then by q -spherical completeness of (X, d) , we have that

$$\bigcap_{C_a \in \mathcal{A}_1} C_a = C \neq \emptyset.$$

Here again, it is clear that $C \cap f(X) \neq \emptyset$.

Let $fb \in C$ and $C_a \in \mathcal{A}_1$. Then we have

$$d(fa, fb) \leq \beta_a \text{ and } d(fb, fa) \leq \beta_a.$$

Let us choose $u \in Ta$ such that $d^s(fa, u) = d^s(fa, Ta)$. Notice that this is possible since the map $: Ta \rightarrow \mathbb{R}$ defined by $u \mapsto d(x, u)$ is uniformly continuous with respect to the usual metric on \mathbb{R} .

With $u \in Ta$ satisfying the above condition and $fb \in C$, we have

$$\begin{aligned} d(fb, Tb) &= \inf\{d(fb, c) : c \in Tb\} \\ &\leq \max\{d(fb, fa), d(fa, u), \inf\{d(u, c) : c \in Tb\}\} \\ &\leq \max\{\beta_a, H(Ta, Tb)\} \\ &< \max\{\beta_a, d(Ta, fa), d(fb, Tb), d(fa, fb)\} \\ &= \max\{\beta_a, d(fb, Tb)\} \end{aligned}$$

which is possible only when $d(fb, Tb) < \beta_a$.

By a similar computation, we have that $d(Tb, fb) < \max\{\beta_a, d(Tb, fb)\}$ which is possible only when $d(Tb, fb) < \beta_a$.

Let now $x \in C_b$. Then

$$d(fb, x) \leq \beta_b < \beta_a ; \text{ and } d(x, fb) \leq \beta_b < \beta_a.$$

We have now that:

$$\begin{aligned} d(fa, x) &\leq \max\{d(fa, fb), d(fb, x)\} \\ &\leq \beta_a \end{aligned}$$

which means that $x \in C_d(fa, \beta_a)$. We have thus shown that

$$C_d(fb, \beta_b) \subseteq C_d(fa, \beta_a). \quad (10)$$

Similarly we can show that

$$C_{d-1}(fb, \beta_b) \subseteq C_{d-1}(fa, \beta_a). \quad (11)$$

Equations (10) and (11) imply that for all $C_a \in \mathcal{A}_1$, $C_b \subseteq C_a$. In other words, this says that $C_a \preceq C_b$ for all $C_a \in \mathcal{A}_1$. Thus C_b is an upper bound in \mathcal{A} for the chain \mathcal{A}_1 . We therefore conclude by Zorn's lemma that \mathcal{A} has a maximal element, say, C_z , $z \in X$. We shall prove that $fz \in Tz$. We do this by contradiction.

Suppose on the contrary that $fz \notin Tz$. Then from (6) there exists $fz^* \in Tz$, $fz^* \neq fz$, such that $d^s(fz, fz^*) = d^s(fz, Tz)$.

$$\begin{aligned} d(fz^*, Tz^*) &\leq H(Tz, Tz^*) \\ &< \max\{d(fz, fz^*), d(Tz, fz), d(fz^*, Tz^*)\} \\ &\leq \{\beta_z, d(fz^*, Tz^*)\} \end{aligned}$$

which is possible only if $d(fz^*, Tz^*) < \beta_z$.

Similarly we have that $d(Tz^*, fz^*) < \max\{\beta_z, d(Tz^*, fz^*)\}$ which is possible only if $d(Tz^*, fz^*) < \beta_z$.

Let $y \in C_{z^*}$. Then

$$d(y, fz^*) \leq \beta_{z^*} < \beta_z ; \text{ and } d(fz^*, y) \leq \beta_{z^*} < \beta_z.$$

Also

$$d(y, fz) \leq \max\{d(y, fz^*), d(fz^*, fz)\} \leq \beta_z$$

and

$$d(fz, y) \leq \max\{d(fz, fz^*), d(fz^*, y)\} \leq \beta_z.$$

This just means that y belongs to the ball C_z , so that $C_{z^*} \subseteq C_z$.

The inequality $d^s(fz, fz^*) = d^s(fz, Tz) > d^s(fz^*, Tz^*)$ implies that z does not belong to the ball C_{z^*} and this implies that $C_{z^*} \subsetneq C_z$ which contradicts the maximality of C_z . So we must have that $fz \in Tz$.

Further, assume (8), and (9). From (8), we have

$$d(fz, f^2z) \leq H(Tfz, Tfz) = 0,$$

and

$$d(f^2z, f^2z) \leq H(Tfz, Tfz) = 0.$$

This means that $d(fz, f^2z) = 0 = d(f^2z, f^2z)$, hence $f^2z = fz$. From (9), $fz = f^2z \in fTz \subseteq Tfz$. Thus fz is a common fixed point of f and T .

We shall now prove uniqueness.

Suppose there is z^* such that $fz^* \neq fz$ and $fz^* = f^2z^* \in Tfz^*$. From (7), (8), we have

$$\begin{aligned} d(fz, fz^*) &= d(f^2z, f^2z^*) \leq H(Tf^2z, Tfz^*) \\ &\leq H(Tfz, Tfz^*) \\ &< \max\{d(f^2z, f^2z^*), d(f^2z, Tfz), d(f^2z^*, Tfz^*)\} \\ &= d(fz, fz^*) \end{aligned}$$

and

$$\begin{aligned}
d(fz^*, fz) &= d(f^2z^*, f^2z) \leq H(Tf^2z^*, Tfz) \\
&\leq H(Tfz^*, Tfz) \\
&< \max\{d(f^2z^*, f^2z), d(f^2z^*, Tfz^*), d(f^2z, Tfz)\} \\
&= d(fz^*, fz).
\end{aligned}$$

This implies that $fz^* = fz$. Thus $z = f^2z$ is the unique common fixed point of f and T . □

Remark 4.5 If $f = I$ (Identity map), then the first part of Theorem 4.4 is the main theorem of Agyingi [9].

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